Rastru grafikas algoritmi

Elipses līnijas veidošanas algoritms

Ellipse-Generating Algorithms

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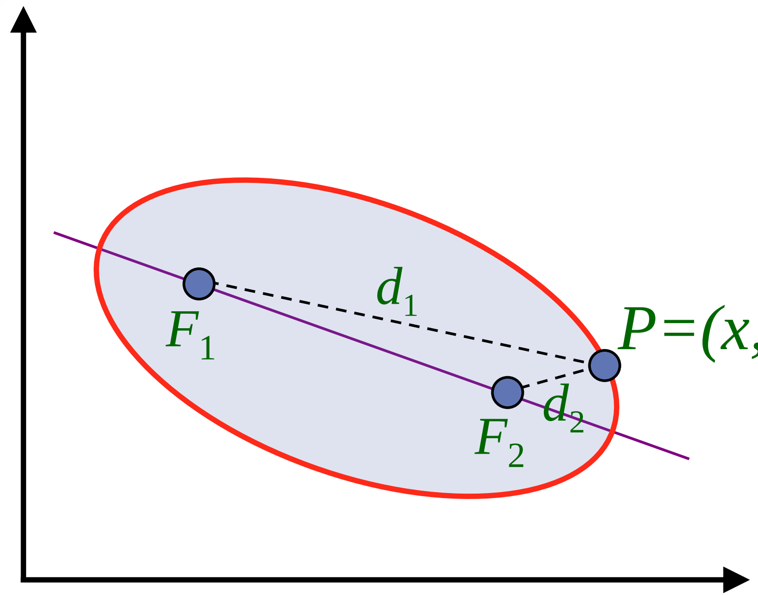


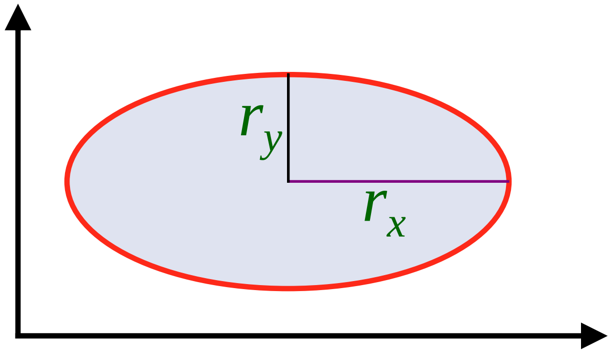
- The sum of the two distances d_1 and d_2, between the fixed positions F_1 and F_2 (called the foci of the ellipse) to any point P on the ellipse, is the same value, i.e.

$$d_1 + d_2 = \mathbf{const}$$

Ellipse Properties

Expressing distances d_1 and d_2 in terms of the focal coordinates $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$, we have:

$$\sqrt{(x-x_1)^2+(y-y_1)^2}+\sqrt{(x-x_2)^2+(y-y_2)^2}=\text{constant}$$

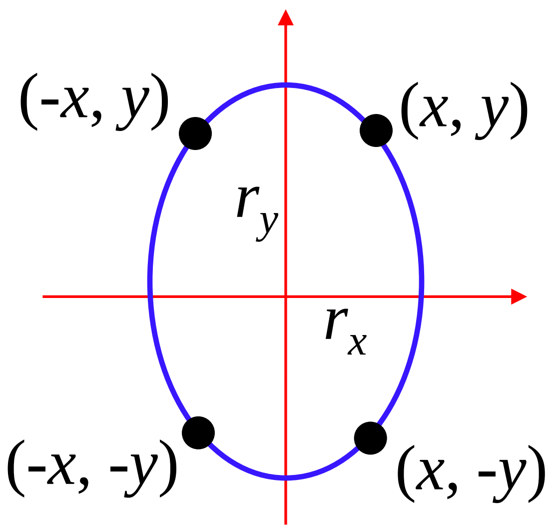


Cartesian coordinates: $$\left(\frac{x-x_c}{r_x}\right)^2+\left(\frac{y-y_c}{r_y}\right)^2=1$$

Polar coordinates: $$x = x_c + r_x \cos\theta$$
$$y = y_c + r_y \sin\theta$$

Ellipse Algorithms

- Symmetry between quadrants
- Not symmetric between the two octants of a quadrant
- Thus, we must calculate pixel positions along the elliptical arc through one quadrant and then we obtain positions in the remaining 3 quadrants by symmetry

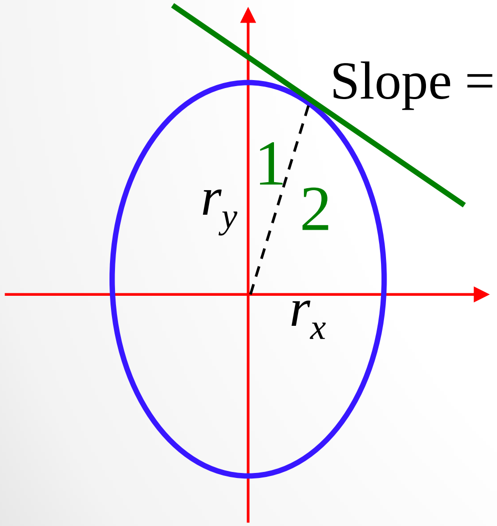


Ellipse Algorithms

$$f_{ellipse}(x,y) = r_y^2 x^2 + r_x^2 y^2 - r_x^2 r_y^2$$

Decision parameter:

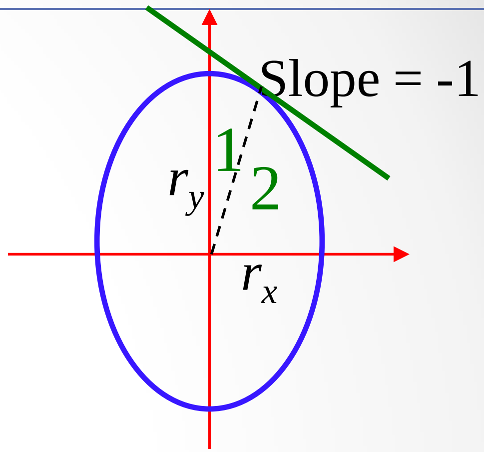
$$f_{ellipse}(x,y) = \begin{cases} < 0 & \text{if } (x,y) \text{ is inside the ellipse} \\ = 0 & \text{if } (x,y) \text{ is on the ellipse} \\ > 0 & \text{if } (x,y) \text{ is outside the ellipse} \end{cases}$$



$$Slope = \frac{dy}{dx} = -\frac{2r_y^2 x}{2r_x^2 y}$$

Ellipse Algorithms

- Starting at $(0, r_y)$ we take unit steps in the x direction until we reach the boundary between region 1 and region 2. Then we take unit steps in the y direction over the remainder of the curve in the first quadrant.



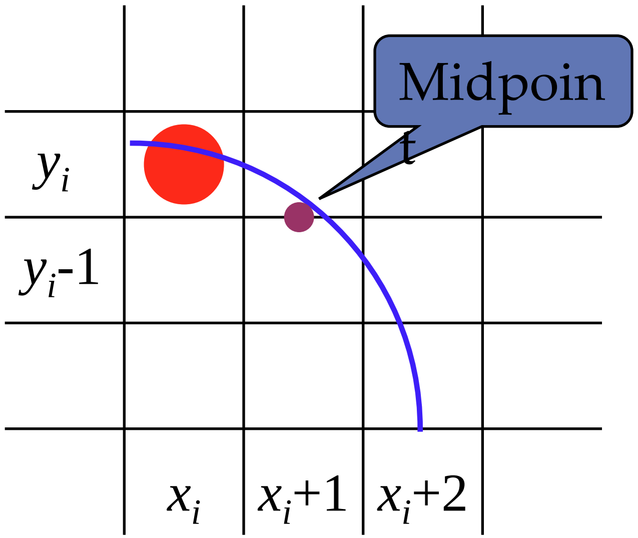
- At the boundary

$$\frac{dy}{dx} = -1 \quad \Rightarrow \quad 2r_y^2 x = 2r_x^2 y$$

- therefore, we move out of region 1 whenever

$$2r_y^2 x \geq 2r_x^2 y$$

Midpoint Ellipse Algorithm



Assuming that we have just plotted the pixels at **(x_i, y_i)**.
The next position is determined by:

$$p1_i = f_{ellipse}(x_i + 1, y_i - \tfrac{1}{2})$$

$$= r_y^2 (x_i + 1)^2 + r_x^2 (y_i - \tfrac{1}{2})^2 - r_x^2 r_y^2$$

If $p1_i < 0$ the midpoint is inside the ellipse $\Rightarrow$ **y_i** is closer
If $p1i \geq 0$ the midpoint is outside the ellipse $\Rightarrow$ **$y_i - 1$** is closer

Decision Parameter (Region 1)

At the next position $[x_{i+1} + 1 = x_i + 2]$

$$p1_{i+1} = f_{ellipse}(x_{i+1} + 1, y_{i+1} - \tfrac{1}{2})$$
$$= r_y^2 (x_i + 2)^2 + r_x^2 (y_{i+1} - \tfrac{1}{2})^2 - r_x^2 r_y^2$$

OR

$$p1_{i+1} = p1_i + 2r_y^2 (x_i + 1)^2 + r_y^2 + r_x^2 \left[(y_{i+1} - \tfrac{1}{2})^2 - (y_i - \tfrac{1}{2})^2 \right]$$

where $y_{i+1} = y_i$ or $y_{i+1} = y_i - 1$

Decision Parameter (Region 1)

Decision parameters are incremented by:

$$increment = \begin{cases} 2r_y^2 x_{i+1} + r_y^2 & \text{if } p1_i < 0 \\ 2r_y^2 x_{i+1} + r_y^2 - 2r_x^2 y_{i+1} & \text{if } p1_i \geq 0 \end{cases}$$

Use only addition and subtraction by obtaining

$$2r_y^2 x \quad \text{and} \quad 2r_x^2 y$$

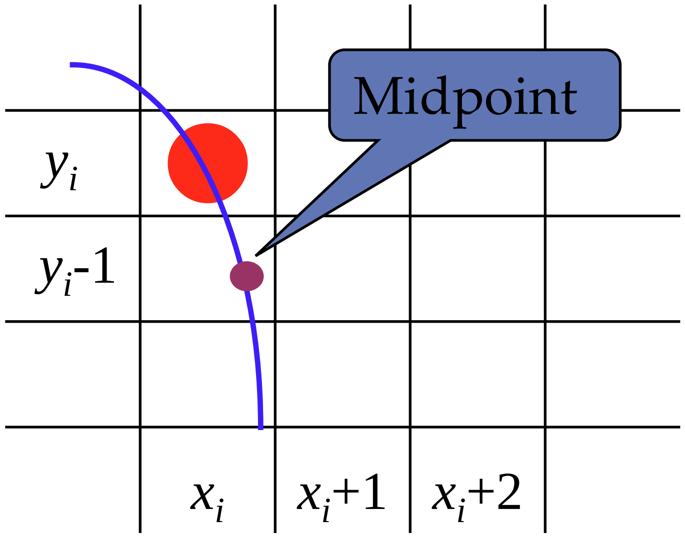
At initial position **$(0, r_y)$**

$$2r_y^2 x = 0$$
$$2r_x^2 y = 2r_x^2 r_y$$

$$\begin{aligned} p1_0 &= f_{ellipse}(1, r_y - \tfrac{1}{2}) = r_y^2 + r_x^2 (r_y - \tfrac{1}{2})^2 - r_x^2 r_y^2 \\ &= r_y^2 - r_x^2 r_y + \tfrac{1}{4} r_x^2 \end{aligned}$$

Region 2

Over region 2, step in the negative y direction and midpoint is taken between horizontal pixels at each step.



Decision parameter:

$$p2_i = f_{ellipse}(x_i + \tfrac{1}{2}, y_i - 1)$$

$$= r_y^2(x_i + \tfrac{1}{2})^2 + r_x^2(y_i - 1)^2 - r_x^2 r_y^2$$

If $p2_i > 0$ the midpoint is outside the ellipse $\Rightarrow$ $\boldsymbol{x_i}$ is closer

If $p2i \leq 0$ the midpoint is inside the ellipse $\Rightarrow$ $x_i + 1$ is closer

Decision Parameter (Region 2)

At the next position $[y_{i+1} - 1 = y_i - 2]$

$$p2_{i+1} = f_{ellipse}(x_{i+1} + \tfrac{1}{2}, y_{i+1} - 1)$$
$$= r_y^2 (x_{i+1} + \tfrac{1}{2})^2 + r_x^2 (y_i - 2)^2 - r_x^2 r_y^2$$

OR

$$p2_{i+1} = p2_i - 2r_x^2 (y_i - 1) + r_x^2 + r_y^2 \left[(x_{i+1} + \tfrac{1}{2})^2 - (x_i + \tfrac{1}{2})^2 \right]$$

where $x_{i+1} = x_i$ or $x_{i+1} = x_i + 1$

Decision Parameter (Region 2)

Decision parameters are incremented by:

$$increment = \begin{cases} -2r_x^2 y_{i+1} + r_x^2 & \text{if } \ p2_i > 0 \\ 2r_y^2 x_{i+1} - 2r_x^2 y_{i+1} + r_x^2 & \text{if } \ p2_i \leq 0 \end{cases}$$

At initial position **(x_0, y_0)** is taken at the last position selected in region 1

$$p2_0 = f_{ellipse}(x_0 + \tfrac{1}{2}, y_0 - 1)$$

$$= r_y^2 (x_0 + \tfrac{1}{2})^2 + r_x^2 (y_0 - 1)^2 - r_x^2 r_y^2$$

Midpoint Ellipse Algorithm

1. Input r_x, r_y, and ellipse center (x_c, y_c), and obtain the first point on an ellipse centered on the origin as

$$(x_0, y_0) = (0, r_y)$$

2. Calculate the initial parameter in region 1 as

$$p1_0 = r_y^2 - r_x^2 r_y + \tfrac{1}{4} r_x^2$$

3. At each x_i position, starting at $i = 0$, if $p1_i < 0$, the next point along the ellipse centered on (0, 0) is $(x_i + 1, y_i)$ and

$$p1_{i+1} = p1_i + 2r_y^2 x_{i+1} + r_y^2$$

otherwise, the next point is $(x_i + 1, y_i - 1)$ and

$$p1_{i+1} = p1_i + 2r_y^2 x_{i+1} - 2r_x^2 y_{i+1} + r_y^2$$

and continue until $2r_y^2 x \geq 2r_x^2 y$

Midpoint Ellipse Algorithm

4. (x_0, y_0) is the last position calculated in region 1.
   Calculate the initial parameter in region 2 as
   $$p2_0 = r_y^2(x_0 + \tfrac{1}{2})^2 + r_x^2(y_0 - 1)^2 - r_x^2 r_y^2$$
5. At each y_i position, starting at $i = 0$, if $p2_i > 0$, the next point along the ellipse centered on (0, 0) is $(x_i, y_i - 1)$ and
   $$p2_{i+1} = p2_i - 2r_x^2 y_{i+1} + r_x^2$$
   otherwise, the next point is $(x_i + 1, y_i - 1)$ and
   $$p2_{i+1} = p2_i + 2r_y^2 x_{i+1} - 2r_x^2 y_{i+1} + r_x^2$$
   Use the same incremental calculations as in region 1.
   Continue until $y = 0$.
6. For both regions determine symmetry points in the other three quadrants.
7. Move each calculated pixel position (x, y) onto the elliptical path centered on (x_c, y_c) and plot the coordinate values

   $$\mathbf{x = x + x_c}, \qquad \mathbf{y = y + y_c}$$

Example

$r_x = 8$, $r_y = 6$

$2r_y^2x = 0$ (with increment $2r_y^2 = 72$)

$2r_x^2y = 2r_x^2r_y$ (with increment $-2r_x^2 = -128$)

i	p_i	x_{i+1}, y_{i+1}	$2r_y^2x_{i+1}$	$2r_x^2y_{i+1}$
0	-332	(1, 6)	72	768
1	-224	(2, 6)	144	768
2	-44	(3, 6)	216	768
3	208	(4, 5)	288	640
4	-108	(5, 5)	360	640
5	288	(6, 4)	432	512
6	244	(7, 3)	504	384

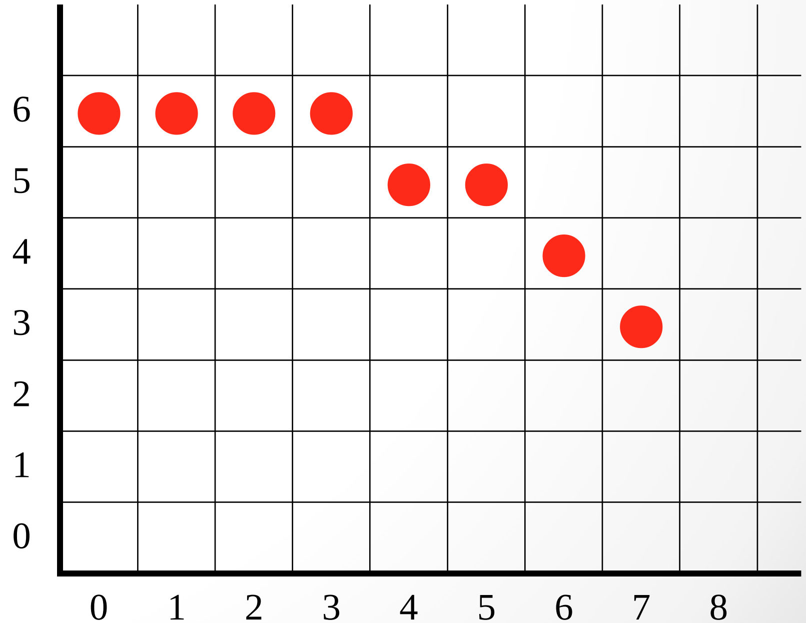
Region 1

$(x_0, y_0) = (0, 6)$

$$p1_0 = r_y^2 - r_x^2 r_y + \tfrac{1}{4} r_x^2 = -332$$

Move out of region 1 since

$2r_y^2x > 2r_x^2y$



Example

Region 2

$(x_0, y_0) = (7, 3)$ (Last position in region 1)

$$p2_0 = f_{ellipse}(7+\tfrac{1}{2}, 2) = -151$$

i	p_i	x_{i+1}, y_{i+1}	$2r_y^2 x_{i+1}$	$2r_x^2 y_{i+1}$
0	-151	(8, 2)	576	256
1	233	(8, 1)	576	128
2	745	(8, 0)	-	-

Stop at $y = 0$